MATHEMATICAL COMMUNICATIONS 111
Math. Commun. **19**(2014), 111–118

On two Thomae-type transformations for hypergeometric series with integral parameter differences

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Received September 9, 2013; accepted January 15, 2014

Abstract. We obtain two new Thomae-type transformations for hypergeometric series with r pairs of numeratorial and denominatorial parameters differing by positive integers. This is achieved by application of the so-called Beta integral method developed by Krattenthaler and Rao [Symposium on *Symmetries in Science* (ed. B. Gruber), Kluwer (2004)] to two recently obtained Euler-type transformations. Some special cases are given.

AMS subject classifications: 33C15, 33C20

Key words: generalized hypergeometric series, Thomae transformations, generalized Euler-type transformations

1. Introduction

The generalized hypergeometric function ${}_pF_q(x)$ is defined for complex parameters and argument by the series

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{x^k}{k!}. \tag{1} $$

When $q \geq p$, this series converges for $|x| < \infty$, but when $q = p - 1$, convergence occurs when $|x| < 1$ (unless the series terminates). In (1), the Pochhammer symbol or ascending factorial $(a)_n$ is given for integer n by

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n = 0) \\ a(a+1)\ldots(a+n-1) & (n \geq 1), \end{cases} $$

where Γ is the gamma function. In what follows we shall adopt the convention of writing the finite sequence of parameters $(a_1, a_2, \ldots, a_p)$ simply by (a_p) and the product of p Pochhammer symbols by

$$ ((a_p))_k \equiv (a_1)_k \ldots (a_p)_k, $$

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where an empty product $p = 0$ is interpreted as unity.

Recent work has been carried out on the extension of various summations theorems, such as those of Gauss, Kummer, Bailey and Watson [1, 6, 7], and also of Euler-type transformations to higher-order hypergeometric functions with r pairs of numeratorial and denominatorial parameters differing by positive integers [3, 4]. Our interest in this note is concerned with obtaining similar extensions of the two-term Thomae transformation [8, p. 52]

$$ {}_3F_2\left[\begin{matrix}a,\, b,\, c\\ d,\, e\end{matrix};1\right] = \frac{\Gamma(d)\Gamma(e)\Gamma(\sigma)}{\Gamma(a)\Gamma(b+\sigma)\Gamma(c+\sigma)}\, {}_3F_2\left[\begin{matrix}c-a,\, d-a,\, \sigma\\ b+\sigma,\, c+\sigma\end{matrix};1\right] $$

for $\Re(\sigma) > 0$, $\Re(a) > 0$, where $\sigma = e + d - a - b - c$ is the parametric excess. Many other results of the above type, including three-term Thomae transformations, are given in [8, pp. 116-121]; see also [9].

The so-called Beta integral method introduced by Krathenthaler and Rao [2] generates new identities for hypergeometric series for some fixed value of the argument (usually 1) from known identities for hypergeometric series with a smaller number of parameters involving the argument x, $1-x$ or a combination of their powers. The basic idea of this method is to multiply the known hypergeometric identity by the factor $x^{d-1}(1-x)^{e-d-1}$, where e and d are suitable parameters, integrate term by term over $[0, 1]$ making use of the beta integral representation

$$ \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \qquad (\Re(a) > 0,\ \Re(b) > 0) \tag{2} $$

and finally to rewrite the result in terms of a new hypergeometric series. We apply this method to two Euler-type transformations obtained recently in [3, 4] to derive two two-term Thomae-type transformations for hypergeometric functions with r pairs of numeratorial and denominatorial parameters differing by positive integers.

2. Extended Thomae-type transformations

Our starting point is the following Euler-type transformations for hypergeometric functions with r pairs of numeratorial and denominatorial parameters differing by positive integers (m_r).

Theorem 1. *Let (m_r) be a sequence of positive integers with $m := m_1 + \cdots + m_r$. Then we have the two Euler-type transformations [3, 4] for $|\arg(1-x)| < \pi$*

$$ {}_{r+2}F_{r+1}\left[\begin{matrix}a, b,\, (f_r + m_r)\\ c,\quad (f_r)\end{matrix};x\right] = (1-x)^{-a}\, {}_{m+2}F_{m+1}\left[\begin{matrix}a,\, c{-}b{-}m,\, (\xi_m + 1)\\ c,\quad (\xi_m)\end{matrix};\frac{x}{x-1}\right] \tag{3} $$

provided $b \neq f_j$ $(1 \leq j \leq r)$, $(c-b-m)_m \neq 0$ and

$$ {}_{r+2}F_{r+1}\left[\begin{matrix}a, b,\, (f_r + m_r)\\ c,\quad (f_r)\end{matrix};x\right] = (1-x)^{c-a-b-m}\, {}_{m+2}F_{m+1}\left[\begin{matrix}c{-}a{-}m,\, c{-}b{-}m,\, (\eta_m + 1)\\ c,\quad (\eta_m)\end{matrix};x\right] \tag{4} $$

provided $(c-a-m)_m \neq 0$, $(c-b-m)_m \neq 0$. *The* (ξ_m) *and* (η_m) *are respectively the nonvanishing zeros of the associated parametric polynomials* $Q_m(t)$ *and* $\hat{Q}_m(t)$ *defined below.*

The parametric polynomials $Q_m(t)$ and $\hat{Q}_m(t)$, both of degree $m = m_1 + \cdots + m_r$, are given by

$$Q_m(t) = \frac{1}{(\lambda)_m} \sum_{k=0}^{m} (b)_k C_{k,r} (t)_k (\lambda - t)_{m-k}, \tag{5}$$

where $\lambda := b - a - m$, and

$$\hat{Q}_m(t) = \sum_{k=0}^{m} \frac{(-1)^k C_{k,r} (a)_k (b)_k (t)_k}{(c-a-m)_k (c-b-m)_k} G_{m,k}(t), \tag{6}$$

where

$$G_{m,k}(t) := {}_3F_2 \left[\begin{matrix} -m+k, t+k, c-a-b-m \\ c-a-m+k, c-b-m+k \end{matrix} ; 1 \right].$$

The coefficients $C_{k,r}$ are defined for $0 \leq k \leq m$ by

$$C_{k,r} = \frac{1}{\Lambda} \sum_{j=k}^{m} \sigma_j \mathbf{S}_j^{(k)}, \qquad \Lambda = (f_1)_{m_1} \ldots (f_r)_{m_r}, \tag{7}$$

with $C_{0,r} = 1$, $C_{m,r} = 1/\Lambda$. The $\mathbf{S}_j^{(k)}$ denote the Stirling numbers of the second kind and the σ_j $(0 \leq j \leq m)$ are generated by the relation

$$(f_1 + x)_{m_1} \cdots (f_r + x)_{m_r} = \sum_{j=0}^{m} \sigma_j x^j. \tag{8}$$

For $0 \leq k \leq m$, the function $G_{m,k}(t)$ is a polynomial in t of degree $m-k$ and both $Q_m(t)$ and $\hat{Q}_m(t)$ are normalized so that $Q_m(0) = \hat{Q}_m(0) = 1$.

Remark 1. *In [5], an alternative representation for the coefficients* $C_{k,r}$ *is given as the terminating hypergeometric series of unit argument*

$$C_{k,r} = \frac{(-1)^k}{k!} {}_{r+1}F_r \left[\begin{matrix} -k, (f_r + m_r) \\ (f_r) \end{matrix} ; 1 \right].$$

When $r = 1$, *with* $f_1 = f$, $m_1 = m$, *Vandermonde's summation theorem [8, p. 243] can be used to show that*

$$C_{k,1} = \binom{m}{k} \frac{1}{(f)_k}. \tag{9}$$

We first apply the Beta integral method [2] to the result in (4) to obtain a new hypergeometric identity. Multiplying both sides by $x^{d-1}(1-x)^{e-d-1}$, where e, d are arbitrary parameters satisfying $\Re(e-d) > 0$, $\Re(d) > 0$, we integrate over the interval $[0,1]$. The left-hand side yields

$$\int_0^1 x^{d-1}(1-x)^{e-d-1}{}_{r+2}F_{r+1}\left[\begin{matrix} a,\, b,\, (f_r+m_r) \\ c, \quad (f_r)\end{matrix};x\right]dx$$

$$= \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!}\frac{((f_r+m_r))_k}{((f_r))_k}\int_0^1 x^{d+k-1}(1-x)^{e-d-1}dx$$

$$= \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!}\frac{((f_r+m_r))_k}{((f_r))_k}\frac{\Gamma(d+k)\Gamma(e-d)}{\Gamma(e+k)}$$

$$= \frac{\Gamma(d)\Gamma(e-d)}{\Gamma(e)}{}_{r+3}F_{r+2}\left[\begin{matrix} a,\, b,\, d,\, (f_r+m_r) \\ c,\, e, \quad (f_r)\end{matrix};1\right], \tag{10}$$

upon evaluation of the integral by (2) and use of the definition (1) when it is supposed that $\Re(s)>0$, where s is the parametric excess given by

$$s := c+e-a-b-d-m. \tag{11}$$

Proceeding in a similar manner with the right-hand side of (4), we obtain

$$\int_0^1 x^{d-1}(1-x)^{s-1}{}_{m+2}F_{m+1}\left[\begin{matrix} c-a-m,\, c-b-m,\, (\eta_m+1) \\ c, \quad (\eta_m)\end{matrix};x\right]dx$$

$$= \sum_{k=0}^{\infty} \frac{(c-a-m)_k(c-b-m)_k}{(c)_k\, k!}\frac{((\eta_m+1))_k}{((\eta_m))_k}\int_0^1 x^{d+k-1}(1-x)^{s-1}dx$$

$$= \frac{\Gamma(d)\Gamma(s)}{\Gamma(c+e-a-b-m)}{}_{m+3}F_{m+2}\left[\begin{matrix} c-a-m,\, c-b-m,\, d,\, (\eta_m+1) \\ c,\, c+e-a-b-m, \quad (\eta_m)\end{matrix};1\right]. \tag{12}$$

Then by (10) and (12) we obtain the two-term Thomae-type hypergeometric identity given in the following theorem, where the restriction $\Re(d)>0$ can be removed by appeal to analytic continuation:

Theorem 2. *Let (m_r) be a sequence of positive integers with $m := m_1+\cdots+m_r$. Then*

$${}_{r+3}F_{r+2}\left[\begin{matrix} a,\, b,\, d,\, (f_r+m_r) \\ c,\, e, \quad (f_r)\end{matrix};1\right]$$

$$= \frac{\Gamma(e)\Gamma(s)}{\Gamma(e-d)\Gamma(s+d)}{}_{m+3}F_{m+2}\left[\begin{matrix} c-a-m,\, c-b-m,\, d,\, (\eta_m+1) \\ c,\, s+d, \quad (\eta_m)\end{matrix};1\right] \tag{13}$$

provided $(c-a-m)_m \neq 0$, $(c-b-m)_m \neq 0$, $\Re(e-d)>0$ and $\Re(s)>0$, where s is defined by (11).

The same procedure can be applied to (3) when the parameter $a=-n$ (to ensure convergence of the resulting integral at $x=1$), where n is a non-negative integer, to yield the right-hand side of (3) given by

$$\int_0^1 x^{d-1}(1-x)^{e-d+n-1}{}_{m+2}F_{m+1}\left[\begin{matrix}-n,\, c-b-m,\, (\xi_m+1)\\ c,\qquad (\xi_m)\end{matrix};\frac{x}{x-1}\right]dx$$
$$= \sum_{k=0}^{n}\frac{(-1)^k(-n)_k(c-b-m)_k}{(c)_k\, k!}\,\frac{((\xi_m+1))_k}{((\xi_m))_k}\int_0^1 x^{d+k-1}(1-x)^{e-d+n-k-1}dx$$
$$= \frac{\Gamma(d)\Gamma(e-d+n)}{\Gamma(e+n)}\sum_{k=0}^{n}\frac{(-n)_k(c-b-m)_k(d)_k}{(c)_k(1-e+d-n)_k k!}\,\frac{((\xi_m+1))_k}{((\xi_m))_k}$$
$$= \frac{\Gamma(d)\Gamma(e-d+n)}{\Gamma(e+n)}\,{}_{m+3}F_{m+2}\left[\begin{matrix}-n,\, c-b-m,\, d,\, (\xi_m+1)\\ c,\, 1-e+a+d,\qquad (\xi_m)\end{matrix};1\right] \tag{14}$$

provided $\Re(e-d) > 0$, $\Re(d) > 0$. From (10) and (14), and appeal to analytic continuation to remove the restriction $\Re(d) > 0$, we then obtain the finite Thomae-type transformation

Theorem 3. *Let (m_r) be a sequence of positive integers with $m := m_1 + \cdots + m_r$. Then, for non-negative integer n*

$${}_{r+3}F_{r+2}\left[\begin{matrix}-n,\, b,\, d,\, (f_r+m_r)\\ c,\, e,\qquad (f_r)\end{matrix};1\right]$$
$$= \frac{(e-d)_n}{(e)_n}\,{}_{m+3}F_{m+2}\left[\begin{matrix}-n,\, c-b-m,\, d,\, (\xi_m+1)\\ c,\, 1-e+d-n,\qquad (\xi_m)\end{matrix};1\right] \tag{15}$$

provided $b \neq f_j$ $(1 \le j \le r)$, $(c-b-m)_m \neq 0$ and $\Re(e-d) > 0$.

3. Examples

When $r = 0$ (with $m = 0$), from (13) and (15) we recover the known results [9]

$${}_3F_2\left[\begin{matrix}a,\, b,\, d,\\ c,\, e,\end{matrix};1\right] = \frac{\Gamma(e)\Gamma(c+e-a-b-d)}{\Gamma(e-d)\Gamma(c+e-a-b)}\,{}_3F_2\left[\begin{matrix}c-a,\, c-b,\, d\\ c,\, c+e-a-b\end{matrix};1\right]$$

for $\Re(e-d) > 0$, $\Re(e+c-a-b-d) > 0$ and

$${}_3F_2\left[\begin{matrix}-n,\, b,\, d,\\ c,\, e,\end{matrix};1\right] = \frac{(e-d)_n}{(e)_n}\,{}_3F_2\left[\begin{matrix}-n,\, c-b,\, d,\\ c,\, 1-e+d-n,\end{matrix};1\right]$$

for $\Re(e-d) > 0$ with n a non-negative integer.

In the particular case $r = 1$, $m_1 = m = 1$, $f_1 = f$, we have the parametric polynomial from (5)

$$Q_1(t) = 1 + \frac{(b-f)t}{(c-b-1)f}$$

with the nonvanishing zero $\xi_1 = \xi$ (provided $b \neq f$, $c-b-1 \neq 0$) given by

$$\xi = \frac{(c-b-1)f}{f-b}, \tag{16}$$

and from (6)

$$\hat{Q}_1(t) = 1 - \frac{\{(c-a-b-1)f + ab\}t}{(c-a-1)(c-b-1)f}$$

with the nonvanishing zero $\eta_1 = \eta$ (provided $c-a-1 \neq 0$, $c-b-1 \neq 0$) given by

$$\eta = \frac{(c-a-1)(c-b-1)f}{ab + (c-a-b-1)f}. \tag{17}$$

Then from (13) and (15) we have the transformations

$$ {}_4F_3\left[\begin{matrix} a,\ b,\ d,\ f+1 \\ c,\ e,\quad f \end{matrix};1\right] = \frac{\Gamma(e)\Gamma(s)}{\Gamma(e-d)\Gamma(s+d)}\, {}_4F_3\left[\begin{matrix} c-a-1,\ c-b-1,\ d,\ \eta+1 \\ c,\ s+d,\quad \eta \end{matrix};1\right]$$

provided $c-a-1 \neq 0$, $c-b-1 \neq 0$, $\Re(e-d) > 0$ and $\Re(s) > 0$, where s is defined by (11) with $m = 1$, and

$$ {}_4F_3\left[\begin{matrix} -n,\ b,\ d,\ f+1 \\ c,\ e,\quad f \end{matrix};1\right] = \frac{(e-d)_n}{(e)_n}\, {}_4F_3\left[\begin{matrix} -n,\ c-b-1,\ d,\ \xi+1 \\ c,\ 1-e+d-n,\quad \xi \end{matrix};1\right]$$

for non-negative integer n and $\Re(e-d) > 0$.

In the case $r = 1$, $m_1 = 2$, $f_1 = f$, we have $C_{0,r} = 1$, $C_{1,r} = 2/f$ and $C_{2,r} = 1/(f)_2$ by (9). From (5) and (6) we obtain after a little algebra the quadratic parametric polynomials $Q_2(t)$ (with zeros ξ_1 and ξ_2) and $\hat{Q}_2(t)$ (with zeros η_1 and η_2) given by

$$Q_2(t) = 1 - \frac{2(f-b)t}{(c-b-2)f} + \frac{(f-b)_2 t(t+1)}{(c-b-2)_2 (f)_2}$$

and

$$\hat{Q}_2(t) = 1 - \frac{2Bt}{(c-a-2)(c-b-2)} + \frac{Ct(1+t)}{(c-a-2)_2(c-b-2)_2},$$

where

$$B := \sigma' + \frac{ab}{f}, \qquad C := \sigma'(\sigma'+1) + \frac{2ab\sigma'}{f} + \frac{(a)_2(b)_2}{(f)_2}, \qquad \sigma' := c-a-b-2.$$

For example, if $a = \frac{1}{4}$, $b = \frac{5}{2}$, $c = \frac{3}{2}$ and $f = \frac{1}{2}$ we have

$$Q_2(t) = 1 - \tfrac{8}{3}t + \tfrac{4}{9}t(1+t), \qquad \hat{Q}_2(t) = 1 + \tfrac{16}{9}t - \tfrac{68}{27}t(1+t),$$

whence $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{9}{2}$ and $\eta_1 = \frac{1}{2}$, $\eta_2 = -\frac{27}{34}$. The transformations in (13) and (15) then yield

$$ {}_4F_3\left[\begin{matrix} \frac{1}{4},\ \frac{5}{2},\ d,\ \frac{5}{2} \\ \frac{3}{2},\ e,\ \frac{1}{2} \end{matrix};1\right] = \frac{\Gamma(e)\Gamma(e-d-\frac{13}{4})}{\Gamma(e-d)\Gamma(e-\frac{13}{4})}\, {}_4F_3\left[\begin{matrix} -\frac{3}{4},\ -3,\ d,\ \frac{7}{34} \\ e-\frac{13}{4},\ \frac{1}{2},\ -\frac{27}{34} \end{matrix};1\right] \tag{18}$$

provided $\Re(e-d) > \frac{13}{4}$, and

$$ {}_4F_3\left[\begin{matrix} -n,\ \frac{5}{2},\ d,\ \frac{5}{2} \\ \frac{3}{2},\ e,\ \frac{1}{2} \end{matrix};1\right] = \frac{(e-d)_n}{(e)_n}\, {}_4F_3\left[\begin{matrix} -n,\ -3,\ d,\ \frac{11}{2} \\ 1-e+d-n,\ \frac{1}{2},\ \frac{9}{2} \end{matrix};1\right] \tag{19}$$

for non-negative integer n. We remark that a contraction of the order of the hypergeometric functions on the right-hand sides of (18) and (19) has been possible since $c = \xi_1 + 1 = \eta_1 + 1 = \frac{3}{2}$. In addition, both series on the right-hand sides terminate: the first with summation index $k = 3$ and the second with index $k = \min\{n, 3\}$. A final point to mention is that for real parameters a, b, c and f it is possible (when $m \geq 2$) to have complex zeros.

4. Concluding remarks

We have employed the Beta Integral method of Krattenthaler and Rao [2] applied to two recently obtained Euler-type transformations for hypergeometric functions with r pairs of numeratorial and denominatorial parameters differing by positive integers (m_r). By this, we have established two Thomae-type transformations given in Theorems 2 and 3.

In order to write the hypergeometric series in (13) and (15) we require the zeros (η_m) and (ξ_m) of the parametric polynomials $\hat{Q}_m(t)$ and $Q_m(t)$, respectively. However, to evaluate the series on the right-hand sides of (13) and (15), *it is not necessary to evaluate these zeros.* This observation can be understood by reference to the hypergeometric series

$$F \equiv {}_{m+2}F_{m+1}\left[\begin{matrix}\alpha,\ \beta,\ (\xi_m+1)\\ \gamma,\qquad (\xi_m)\end{matrix};1\right] = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k\, k!}\left(1+\frac{k}{\xi_1}\right)\cdots\left(1+\frac{k}{\xi_m}\right)$$

upon use of the fact that $(a+1)_k/(a)_k = 1+(k/a)$. Since the parametric polynomial $Q_m(t)$ in (5) can be written as $Q_m(t) = \prod_{r=1}^{m}\{1-(t/\xi_r)\}$, it follows that

$$F = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k\, k!}\, Q_m(-k).$$

Consequently, it is sufficient to know only the parametric polynomial $Q_m(t)$. A similar remark applies to the series involving the zeros (η_m) with the parametric polynomial $Q_m(-k)$ replaced by $\hat{Q}_m(-k)$.

Acknowledgement:

Y. S. Kim acknowledges the support of the Wonkwang University Research Fund (2014).

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